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Using Cross Efficiency with Symmetric Weights for the Method DEAHP

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ABSTRACT: Data envelopment analysis (DEA) is a useful tool to measure and recognize the effectiveness and performance of decision-making units. On the other hand, analytical hierarchy process (AHP) is a useful tool in the field of multi-criteria decision-making problems (MCDA) and it can be helpful to rank the set of options with distinct criteria and choose the best or most appropriate option among them. In 2004, a hybrid method so-called DEAHP was resulted from AHP and DEA methods proposed or suggested by Ramanatan through which the relative weights of options and criteria can be obtained from the matrix of paired comparisons. However, the weights generated by DEAHP may be quite irrational even wrong for a highly inconsistent pairwise comparison matrix. With a violation example we showed that the proposed model does not preserve the ranking. Therefore, in the current research we use the symmetric weights for a more logical choice of criteria and options.
Keywords: Data Envelopment Analysis, Analytical Hierarchy Process, Weight Restrictions, Symmetric Weights.

ORIGINAL ARTICLE

INTRODUCTION

Data envelopment analysis (DEA) is a useful tool for measuring and identifying the efficiency of decision making units. Data envelopment analysis has been used to determine the best weights through measuring the efficiency and it aims to recognize the efficiency of a system or a decision making unit.

One of the most important limitations in conventional DEA model is excessive weight flexibility, which seeks maximum efficiency by DMU. This process will bring up two problems. First the generated weights are implausible because DEA ignore one or more variable, second, they are unacceptable because the results are inconsistent with expert judgments prepared by DM. The flexibility of conventional DEA models lead to generate illogical weights. Therefore, the weight restrictions are necessary to solve these problems.

On the other hand, analytical hierarchy process (AHP) is an appropriate tool in the field of multi-criteria decision making problems (MCDA) and it can be useful for rating a set of options with different criteria and choosing the most appropriate or best option. Analytical hierarchy process is one of the most comprehensive systems for decision making with multiple criteria.

In 1986, Thomas Saati (the founder of this method) proposed the following four principles as the principles of the analytical hierarchy process and established all calculations, rules and regulations on the basis of these principles.

These principles are:

Principle1: reverse condition: if the preference of element A to element B is equal to n, the preference of element B to element A will be equal to 1/n.

Principle2: homogeneity principle: elements A and B should be homogenous and comparable. In the other words, the preference of element A to element B cannot be infinite or zero.

Principle 3: dependency: each hierarchal element can be related to the element upper than itself and this dependency can be continued up to the highest level.

Principle 4: expectations: whenever a change occurs in a hierarchical structure, the evaluation process must be done again.

Today, analytical hierarchy process (AHP) has been more developed in both areas of theory and practice. Apart from Saaty's eigenvector method (EM) (2000), which is the most widely used priority method, Chu et al. (1979) proposed a weighted least-squares method (WLSM). Crawford (1987) proposed a logarithmic least-squares method (LLSM). Saaty and Vargas (1984) presented a least-squares method (LSM). Cogger and Yu (1985) suggested a gradient eigenweight method (GEM) and a least distance method (LDM). Islei and Lockett (1988) developed a geometric least-squares method (GLSM). Bryson (1995) put forward a goal programming method (GPM). Bryson and Joseph (1999) also brought forward a logarithmic goal programming approach (LGPA) and other alternative approaches

Since these two methods cannot alone provide a good solution to determine the best option, data envelopment analysis is a technique that evaluates the decision making units at the best. In this technique, the previous data or external data is not used as much as possible, but there are situations in which additional information is available or decision makers are willing to enter their personal comments and preferences.

DEAHP, a combination of DEA and AHP methods, introduced by Ramanatan (2006). Uses DEA for determining the relative weights of criteria and options. This method has some significant difficulties. Such as it allows decision making units to choose weights freely, gives the high weights to outputs having strengths, gives the zero weight to outputs that have weaknesses, evaluate the non-logical relative weights to the paired comparison matrixes and also does not maintain the ranking.

The remainder of this paper has the following structure: In Section 1, briefly reviews DEA and DEAHP and cross-efficiency with symmetric weights. In Section2, the recommended method for solving the DEAHP problem. Section 3 illustrates the proposed method using an example. Finally, conclusions are given in Section 4.

BACKGROUND

1) Data envelopment analysis: Defining the principle DEA, in detail, is out of this papers scope and the complete review has given in several Papers for instance in (1978). A brief description of DEA technique is provided in following. Assuming that there are n DMUs each with m inputs and s outputs, the relative efficiency of a particular DMU_0($0 \in \{1,2,...,n\}$) is obtained by solving the following the fractional programming model:

$$\begin{aligned} Max \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} V_i X_{i0}} \\ s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} V_i X_{ij}} \le 1, \qquad j = 1,...,n \\ & u_r \ge 0 \qquad r = 1,...,S \\ & V_i \ge 0 \qquad i = 1,...,m \end{aligned} \tag{1}$$

Where j is DMU index j=1,..., n, x_{ij} (i=1, ..., m) the value of i^{th} input for the j^{th} DMU, y_{rj} (r=1, ..., s) the value of r^{th} output for the j^{th} DMU, u_r (r=1, ..., s) are the weights of outputs, V_i(i=1,..., m) are the weights of inputs and let DMU_0 be a DMU under evaluation. DMU_0 is the efficient if and only if θ_0=1.

The efficiently of DMU_0 is defined as the ratio of weighted its outputs to weighted inputs subjected to the condition that the similar ratios for all DMU_s be lower than or equal to 1. A relative efficiency score of 1 indicates that the DMU under consideration is efficient, whereas a score lower than 1 implies that it is inefficient. This fractional program can be converted into a liner programing problem: if either the denominator or numerator of the ratio is forced to be unity, then the objective function will become linear, and a linear programming problem: if either the denominator or numerator of the ratio is forced to be unity, then the objective function will become linear, and a linear programming problem can be obtained. By setting the denominator of the ratio equal to 1, the reformulated liner programming problem, also known as the multiplier from of CCR input–oriented model, is as follows:

$$\begin{aligned} Max \quad & \theta_0 = \sum_{r=1}^{s} u_r y_{r0} \\ S.t. \quad & \sum_{i=1}^{m} V_i x_{io} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} V_i x_{ij} \le 0, \quad j = 1,...,n \\ & u_r \ge 0 \quad r = 1,...,s \\ & V_i \ge 0 \quad i = 1,...,m \end{aligned} \tag{2}$$

Note that by setting the numerator equal to 1, the multiplier from of CCR output–oriented model is obtained. Because of the nature of formulations, the optimal objective function values of the CCR input and output oriented represent the reciprocal of efficiency scores.

2) DEAHP method

Let

$$A = (a_{ij})_{n \times n} = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \dots & a_{nn} \end{bmatrix} \tag{3}$$

be a pairwise comparison matrix with a_{ii}=1 and $a_{ji}=1/a_{ij}$ for j≠i and W= $(w_1..., w_n)^T$ be its priority vector. The DEAHP views each row of the matrix *A* as a DMU, each column as an output and assumes a dummy input value of one for all the DMU_s. Each DMU has therefore *n* outputs and one dummy constant input, based on which the following input–oriented CCR model 4 is constructed to estimate the local priorities (weights) of the pairwise comparison matrix *A*:

$$\begin{aligned} Max \quad & w_0 = \sum_{j=1}^{n} a_{0j} v_j, \\ s.t. \quad & \begin{cases} u_1 = 1, \\ \sum_{j=1}^{n} a_{ij} v_j - u_1 \le 0, \quad i = 1,...,n, \\ u_1, v_j \ge 0, \quad j = 1,...,n, \end{cases} \end{aligned} \tag{4}$$

Where DMU_0 represents the criterion or alternative under evalution. The optimum objective function value of the above model, w_0^*, represents the DEA efficiency of DMU_0 and is used as its local priority. LP model (4) is solved for all the DMU_s to obtain the local priority vector $W^* = (w_1^*, \ldots, w_n^*)^T$ of the pairwise comparison matrix *A*. It has been proved that the DEAHP can derive true weights if *A* is perfectly consistent, that is, *A* meets the condition of $a_{ij} = a_{ik} a_{kj}$ for all i, j, k = 1, ..., n.

This model is solved for all the DMUs to generate the weight vector $W^* = (w_1^*, \ldots, w_n^*)^T$ of A.

DEAHP offers the following two models to calculate the final weight.

1. It calculates the relative weights of criteria without taking into account the relative weight of criteria.

$$
\begin{aligned}
Max \quad & \sum_{j=1}^{n} u_j w_{oj}^* \\
s.t. \quad & v_1 = 1 \\
& \sum_{j=1}^{n} u_j w_{oj}^* - v_1 \le 0 \qquad i = 1, \ldots, n \\
& v_1 \ge 0,\ u_j \ge 0,\ j = 1, \ldots, n.
\end{aligned} \tag{5}
$$

2. It has calculated the final weight of options considering the relative weight of criteria.

If we want to measure the relative weight of the DEA model, we use the limitations of the assurance area as follows:

$$d_j = \frac{u_j}{u_1}, \quad j = 1, \ldots, n, \qquad \text{s.t.} \qquad d_j = \frac{w_j^*}{w_1^*}, \quad j = 1, \ldots, n,$$

So, the CCR model will be as follow:

$$
\begin{aligned}
Max \quad & u_1 \left(\sum_{j=1}^{n} d_j w_{oj}^*\right) \\
s.t. \quad & v_1 = 1 \\
& u_1 \left(\sum_{j=1}^{n} d_j w_{ij}^*\right) - v_1 \le 0 \qquad i = 1, \ldots, n \\
& v_1 \ge 0,\ u_1 \ge 0.
\end{aligned} \tag{6}
$$

DEAHP method has some disadvantages and the main problem with this approach is that the decision making units are allowed to freely choose the weights and much weight is given to the outputs that have strengths and the zero weight is given to those that have weaknesses; and therefore, unreasonable relative weights are calculated for the paired comparison matrices that to overcome these drawbacks, we propose a new method in which we overcome these problems though applying weight restrictions.

Applying weight restrictions has some problems such as infeasibility of the problem, but we apply these restrictions in such a way that first, the problem remains feasible and second, it chooses the weights symmetrically so that it does not have the problem of DEAHP that is the freely choice of weights.

3) Cross-efficiency with symmetric weights

Dimitrov and Sutton (2010) have presented a model for restricting weights with the goal of promoting symmetry in weight allocation. Then, Jahanshahloo et al. (2011) offer a secondary goal for cross-efficiency evaluation (1986), in which emphasize selecting symmetric weights by DMUs.

One of the most severe limitations of conventional DEA models is their excessive weight flexibility, allowing a DMU to seek maximum efficiency by selecting a mix of weights that either is implausible because it ignores one or more variables, or is unacceptable because it is inconsistent with expert judgment available to the DM. Furthermore, freely selecting weight allocation may lead to two DMUs having equal efficiency scores one with all of its weight on one variable and another with its weight symmetrically allocated to all variables. Thus, this shortcoming has led to the development of weight restrictions DEA models. Allen et al. (1997) considered a lower and upper bound on outputs or inputs as follows:

$$a_r \le u_r y_{rj} \le b_r \quad \forall r, j, \qquad c_i \le v_i x_{ij} \le d_i \quad \forall i, j. \tag{7}$$

But by adding these restrictions to the DEA models, the programs will often be infeasible, and using explicit boundaries for weights is a difficult task. Therefore, Dimitrov and Sutton (2010) have proposed a model that not only has each DMU rating itself as efficient as possible relative to the other DMUs, but also explicitly rewards DMUs that make a symmetric choice of weights. The measure of symmetry is the relative weight of each output dimension to all other output dimensions:

$$\left| u_i y_{oi} - u_j y_{oj} \right| = z_{ij} \quad \forall i, j. \tag{8}$$

z_{ij} in (8) is the difference in symmetry between output dimension i and dimension j for the DMU under evaluation. So, if we minimize the sum of all the Z values ($Z = (z_{ij})$), then we effectively reward symmetry with a symmetry scaling factor $\beta \ge 0$.

Adding the symmetry constraint to the objective function rewrites (2) to

$$\begin{aligned} Max \quad & U^T y_o + Min \ \beta e^T Ze \\ s.t. \quad & V^T x_o = 1 \\ & U^T y_j - V^T x_j \le 0 \qquad j = 1, \dots, n, \\ & |u_i y_{oi} - u_j y_{oj}| = z_{ij} \quad i, j = 1, \dots, n, \\ & U \ge 0, \ V \ge 0 \end{aligned} \tag{9}$$

where e = $(1,1,\dots,1)^T$.

Note that (9) is not linear with the equality constraint. Fortunately, as minimizing $e^T Ze$ may change the equality to ≤ as any optimal solution will have the equality constraint satisfied. With this observation, rewrite (9) to a linear program as:

$$\begin{aligned} \alpha_o = Max \quad & U^T y_o - \beta e^T Ze \\ s.t. \quad & V^T x_o = 1 \\ & U^T y_j - V^T x_j \le 0 \qquad j = 1, \dots, n, \\ & u_i y_{oi} - u_j y_{oj} \le z_{ij} \quad i, j = 1, \dots, n, \\ & -u_i y_{oi} + u_j y_{oj} \le z_{ij} \quad i, j = 1, \dots, n, \\ & U \ge 0, \ V \ge 0 \end{aligned} \tag{10}$$

Clearly, $\alpha_o \in [1, \infty)$. Note that the linear program (10) has the same feasibility region as the linear program (9). Instead of having an explicit bound, we introduce the symmetry scaling factor b as a non-negative importance factor. The above-mentioned subjects can be used for the output-oriented formulation and the BCC model (1984). Also, this approach can be extended based on some preference structures (2010).

The cross-efficiency method with symmetric weights is in the following algorithm (2011).

Step 1: Determine simple efficiencies, $\alpha_o, (o = 1, \dots, n)$ for all DMUs after solving model (10). In this step, we obtain weights as (u_o^*, v_o^*).

Step 2: The cross-efficiency for any DMU_j, using the weights that DMU_o has chosen in model (10), is then $\theta_{oj} = \frac{u_o^* y_j}{v_o^* x_j}, \quad o, j = 1, \dots, n,$

where (*) denotes optimal values in model (10). For $DMU_j \ (j = 1, \dots, n)$, the average of all $\theta_{oj}, \ (o = 1, \dots, n)$, namely $\theta_j = \frac{1}{n}\sum_{o=1}^{n} \theta_{oj}, \ (j = 1, \dots, n)$ is our new cross-efficiency score for DMU_j.

The recommended method for solving the DEAHP problem

In this section we discuss our proposed model. The row elements consider the paired comparisons matrix as the output of DMU and select the consonant virtual input equal to 1 for all DMU to obtain the relative weights of the paired comparisons matrix$A = [a_{ij}]_{n \times n}$.

The multiple forms of CCR model are used as input. Suppose $(\boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{n})\, \boldsymbol{u_j}$ is jth output weight, that is$\boldsymbol{a_{ij}}$, and v_1 is the input weight. In this case, the model will be as follows:

$$\begin{aligned} w_o = Max \quad & \sum_{j=1}^{n} u_j a_{oj} - \beta e^T Ze \\ s.t. \quad & v_1 = 1 \\ & \sum_{j=1}^{n} u_j a_{ij} - v_1 \le 0 \qquad i = 1, \dots, n, \\ & u_i a_{oi} - u_j a_{oj} \le z_{ij} \quad i, j = 1, \dots, n, \\ & -u_i a_{oi} + u_j a_{oj} \le z_{ij} \quad i, j = 1, \dots, n, \\ & v_1 \ge 0, \ u_j \ge 0, \ j = 1, \dots, n. \end{aligned} \tag{11}$$

Using the above model, we obtain the measure of cross efficiency of DMU_o as ρ_o^* that refers to the under evaluation criterion or option. The method of cross efficiency with symmetric weights is solved for all the DMUs to generate the weight vector $\rho^* = (\rho_1^*, \dots, \rho_n^*)^T$ of A. DEAHP offers the following two models to calculate the final weight.

1. It calculates the relative weights of criteria without taking into account the relative weight of criteria.

$$\begin{aligned} Max \quad & \sum_{j=1}^{n} u_j \rho_{oj}^* - \beta e^T Ze \\ s.t. \quad & v_1 = 1 \\ & \sum_{j=1}^{n} u_j \rho_{ij}^* - v_1 \le 0 \qquad i = 1, \dots, n, \\ & u_i \rho_{oi}^* - u_j \rho_{oj}^* \le z_{ij} \quad i, j = 1, \dots, n, \\ & -u_i \rho_{oi}^* + u_j \rho_{oj}^* \le z_{ij} \quad i, j = 1, \dots, n, \\ & v_1 \ge 0, \ u_j \ge 0, \ j = 1, \dots, n. \end{aligned} \tag{12}$$

2. It has calculated the final weight of options considering the relative weight of criteria. If we want to measure the relative weight of the DEA model, we use the limitations of the assurance area as follows:

$$d_j = \frac{u_j}{u_1}, \quad j = 1, \dots, n, \qquad s.t. \qquad d_j = \frac{\rho_j^*}{\rho_1^*}, \quad j = 1, \dots, n,$$

So, the CCR model will be as follow:

$$\begin{aligned}
\text{Max} \quad & u_1\left(\sum_{j=1}^{n} d_j \rho_{oj}^{*}\right) \\
\text{s.t.} \quad & v_1 = 1 \\
& u_1\left(\sum_{j=1}^{n} d_j \rho_{ij}^{*}\right) - v_1 \le 0 \qquad i = 1,\ldots,n \\
& v_1 \ge 0,\ u_1 \ge 0.
\end{aligned} \qquad (13)$$

Using two above models, we obtain the measure of cross efficiency of DMU_o as θ_o^* that refers to the under evaluation criterion or option. The method of cross efficiency with symmetric weights is solved for all the DMUs to generate the final weights.

The idea behind models 11 and 12 is to identify optimal weights which restrict by promoting symmetry goal in weight allocation. In these models, we introduce an approach to reward by symmetry selecting weights, a suitable method because of centralization weights on only one variable.

It may be undesirable by funding agencies, financial portfolio selection, etc. So, in the proposed method, we select the measure of cross efficiency with symmetry weights to the under evaluation criterion or option.

Numerical example

Consider the following pairwise comparison matrix, which is borrowed from Saaty (2000):

$$A = \begin{bmatrix} 1 & 4 & 3 & 1 & 3 & 4 \\ 1/4 & 1 & 7 & 3 & 1/5 & 1 \\ 1/3 & 1/7 & 1 & 1/5 & 1/5 & 1/6 \\ 1 & 1/3 & 5 & 1 & 1 & 1/3 \\ 1/3 & 5 & 5 & 1 & 1 & 3 \\ 1/4 & 1 & 6 & 3 & 1/3 & 1 \end{bmatrix}$$

Results for different methods (Wang et al. (2008)) and result of our method are prepared in Table (1).

Table 1. Weights and ranking orders obtained by different priority methods.

Priority method	w_1	w_2	w_3	w_4	w_5	w_6
EM	0.3208(1)	0.1395(3)	0.0348(6)	0.1285(5)	0.2374(2)	0.1391(4)
WLSM	0.4150(1)	0.0936(5)	0.0348(6)	0.1123(4)	0.2190(2)	0.1253(3)
LSM	0.1845(4)	0.2204(1)	0.0371(6)	0.1504(5)	0.2103(2)	0.1937(3)
LLSM	0.3160(1)	0.1391(4)	0.0360(6)	0.1251(5)	0.2360(2)	0.1477(3)
GLSM	0.3407(1)	0.1205(5)	0.0575(6)	0.1495(3)	0.2013(2)	0.1305(4)
GEM	0.3746(1)	0.1722(3)	0.0275(6)	0.1252(4)	0.2254(2)	0.0751(5)
FPM	0.3492(1)	0.1438(3)	0.0528(6)	0.1232(5)	0.1917(2)	0.1392(4)
CCMA	0.2768(1)	0.1695(3)	0.0295(6)	0.1555(5)	0.2072(2)	0.1615(4)
DEAHP	1.0000(1)	1.0000(1)	0.3333(6)	1.0000(1)	1.0000(1)	1.0000(1)
LP-GFW	0.4042(1)	0.2130(3)	0.0466(6)	0.1793(5)	0.3827(2)	0.2056(4)
Our method	0.9565(1)	0.6158(4)	0.2745(6)	0.9192(2)	0.628(3)	0.5755(5)

CONCLUSION

In this paper we have analyzed main difficulties of the DEAHP and proposed a method of cross-efficiency evaluation with selecting symmetric weights to overcome these problems.

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